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Constrained Fock spaces as Virasoro modules

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Abstract

The method of constrained Hamiltonian systems can be used to reduce Fock modules. It is applied to the Virasoro algebra, where a possibly new realization is found.

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1 Constraints

Consider the Virasoro algebra Vir,

$$[L_m, L_n] \approx (n-m)L_{m+n} - \frac{c}{12}(m^3-m)\delta_{m+n}, \tag{1}$$

where $m, n \in \mathbb{Z}$. The purpose of this note is to show that some representations of Vir can be found by applying the machinery of constrained Hamiltonian systems [1, 3], suitably adapted to the representation theory framework, to standard Fock modules. Some of the unconstrained Fock modules may be new.

Let $\mathfrak{g}$ be a Lie algebra with elements $A, B, ...$, and assume that there is an embedding of $\mathfrak{g}$ into a graded Poisson algebra $C^\infty(\mathcal{P})$, where $\mathcal{P}$ is the phase space. Let $P, R, ..$ label constraints χ_P, which may be bosonic or fermionic; $(-)^P$ denotes the corresponding parity factor. The equations $\chi_P \approx 0$ define

a surface in $C^\infty(\mathcal{P})$, where weak equality (i.e. equality modulo constraints) is denoted by $\approx$. Constraints are second class if the Poisson bracket matrix $C_{PR} = [\chi_P, \chi_R]$ is invertible; otherwise, they are first class and generate a Lie algebra. First class constraints are connected to gauge symmetries, and Hamiltonian systems with first class constraints are known as gauge systems [3]. Assume that all constraints are second class. Then the matrix C_{PR} has an inverse, denoted by Δ^{PR}. Our sign convention is $(-)^R \Delta^{PR} C_{RS} = \delta^P_S$. The Dirac bracket

$$[A, B]^* = [A, B] - (-)^R [A, \chi_P] \Delta^{PR} [\chi_R, B] \tag{2}$$

defines a new graded Poisson bracket which is compatible with the constraints: $[A, \chi_R]^* = 0$ for every $A \in \mathfrak{g}$. Of course, there is no guarantee that the operators A, B still generate the same Lie algebra under the Dirac brackets. A sufficient condition for this is that the constraints are covariant in the sense that $[A, \chi_P] = 0$ for every A. A less restrictive condition is often possible. Usually, the constraints can be divided into two sets $\chi_P = (\Phi_a, \Pi^a)$, such that $[\Phi^a, \Phi_b] \approx 0$. The Φ^a are then first class, and Π_a are gauge conditions. It is now sufficient that $[A, \Phi_a] \approx 0$, because the components of Δ^{PR} that involve Π's on both sides vanish.

The factor space $C^\infty(\mathcal{P})/\mathcal{N}$, where $\mathcal{N}$ is the ideal generated by the constraints, is the algebra of functions on the constraint surface, with Poisson structure given by the Dirac bracket. Quantization amounts to replacing Dirac brackets by graded commutators and normal ordering. In particular, if $\mathfrak{g}$ is the diffeomorphism algebra in one dimension, we obtain reduced Fock representations of Vir.

2 Scalar boson

Consider a bosonic Virasoro primary field a_m of zero conformal weight and its canonical conjugate $a^\dagger_n$ (of weight 1), $m, n \in \mathbb{Z}$.

$$[a^\dagger_m, a_n] = \delta_{m+n}, \qquad [a^\dagger_m, a^\dagger_n] = [a_m, a_n] = 0. \tag{3}$$

Set

$$B_m = a^\dagger_m + M m a_m, \qquad \chi_m = a^\dagger_m - M m a_m, \tag{4}$$

$$K_m = \sum_{r=-\infty}^{\infty} r :a^\dagger_{m-r} a_r: , \qquad L_m = K_m + \lambda(m+1) B_m, \tag{5}$$

where M (interpretable as a mass) and λ are parameters. Then L_m generate a Virasoro algebra with central charge $c = 2 - 24M\lambda^2$. The following formulas are useful in the verification.

$$
\begin{aligned}
[\chi_m, \chi_n] &= -[B_m, B_n] = 2Mm\delta_{m+n}, && (6)\\
[B_m, \chi_n] &= 0, && (7)\\
[K_m, a_n] &= (m+n)a_{m+n}, \qquad [K_m, a_n^\dagger] = na_{m+n}^\dagger, && (8)\\
[K_m, \chi_n] &= n\chi_{m+n}, \qquad [K_m, B_n] = nB_{m+n}. && (9)
\end{aligned}
$$

The modes a_m can be eliminated by imposing the constraints $\chi_m \approx 0$. By virtue of (6), $\chi_0 \approx 0$ is first class. However, (4–5) are independent of a_0, so we can impose the extra constraint $\chi_{\bar{0}} = a_0 \approx 0$, making all constraints second class. The non-zero elements of the Poisson bracket matrix are

$$
\begin{aligned}
C_{mn} &= [\chi_m, \chi_n] = 2Mm\delta_{m+n}, \qquad (m \neq 0)\\
C_{0\bar{0}} &= -C_{\bar{0}0} = [\chi_0, a_0] = 1, && (10)
\end{aligned}
$$

with inverse

$$
\begin{aligned}
\Delta^{mn} &= \frac{-1}{2Mm}\delta_{m+n}, \qquad (m \neq 0)\\
\Delta^{\bar{0}0} &= -\Delta^{0\bar{0}} = 1. && (11)
\end{aligned}
$$

The Dirac brackets are

$$
[a_m^\dagger, a_n^\dagger]^* = 0 - \sum_{rs}(Mm\delta_{m+r})(\frac{-1}{2Mr}\delta_{r+s})(Ms\delta_{s+n}) = -\frac{M}{2}m\delta_{m+n}, \quad (12)
$$

when $m \neq 0$. However, because

$$
[a_0^\dagger, a_0^\dagger]^* = [a_0^\dagger, a_0]^* = [a_0, a_0]^* = 0, \quad (13)
$$

(12) holds for the zero mode as well. Now,

$$
\begin{aligned}
[L_m, \chi_n] &= na_{m+n}^\dagger - Mn(m+n)a_{m+n} = n\chi_{m+n} \approx 0,\\
[L_m, a_0] &= ma_m + \lambda\delta_m \approx \frac{1}{M}a_m^\dagger + \lambda\delta_m. && (14)
\end{aligned}
$$

If we now solve the constraints $\chi_m \approx a_0 \approx 0$ for $a_m^\dagger$, the Virasoro generators take the form

$$
L_m \approx \sum_{r=-\infty}^{\infty} \frac{1}{M} :a_{m-r}^\dagger a_r^\dagger: + 2\lambda(m+1)a_m^\dagger, \quad (15)
$$

where terms involving $a_0^\dagger = \chi_0 \approx 0$ should be skipped. One verifies that (15) generates a Virasoro algebra with central charge $c = 1 - 24M\lambda^2$ under the Dirac brackets (12); this is the well-known Feigin-Fuks module ([2, 5] and references therein). Moreover, $a_n^\dagger$ transforms as a one-dimensional Christoffel symbol

$$[L_m, a_n^\dagger]^* = n a_{m+n}^\dagger - M\lambda m(m+1)\delta_{m+n}. \tag{16}$$

3 Scalar fermion

Consider a fermionic Virasoro primary field b_r of weight λ and its canonical conjugate $b_r^\dagger$ (of weight $1-\lambda$). We have

$$\begin{aligned}
\{b_r, b_s^\dagger\} &= \delta_{r+s}, \qquad \{b_r, b_s\} = \{b_r^\dagger, b_s^\dagger\} = 0, && (17)\\
L_m &= -\sum_{r\in\mathbb{Z}+\frac{1}{2}} (-\lambda m + r) :b_{m-r}^\dagger b_r:, && (18)\\
[L_m, b_r] &= ((1-\lambda)m + r) b_{m+r}, \qquad [L_m, b_r^\dagger] = (\lambda m + r) b_{m+r}^\dagger, && (19)
\end{aligned}$$

where anti-commutators are explicitly indicated. For simplicity, we take $r, s \in \mathbb{Z} + 1/2$, so there is no zero mode. The L_m generate a Virasoro algebra with central charge $c = -2(1 - 6\lambda + 6\lambda^2)$. If (and only if) $\lambda = 1/2$ (i.e. $c = 1$), we can eliminate the modes $b_r^\dagger$ by the constraints

$$\chi_r = b_r - b_r^\dagger \approx 0, \tag{20}$$

$$\begin{aligned}
C_{rs} = \{\chi_r, \chi_s\} &= -2\delta_{r+s}, \qquad & \Delta^{rs} &= \frac{1}{2}\delta_{r+s},\\
\{\chi_r, b_s\} &= -\delta_{m+n}, \qquad & \{\chi_r, b_s^\dagger\} &= \delta_{m+n}. \qquad (21)
\end{aligned}$$

The Dirac brackets are

$$\{b_r, b_s\}^* = 0 + \sum_{kl} (-\delta_{r+k})(\frac{1}{2}\delta_{k+l})(-\delta_{l+s}) = \frac{1}{2}\delta_{r+s} \tag{22}$$

From

$$[L_m, \chi_r] = (\frac{m}{2} + r)\chi_{m+r} \approx 0, \tag{23}$$

it follows that

$$L_m = -\sum_{r\in\mathbb{Z}+\frac{1}{2}} (-\frac{m}{2} + r) :b_{m-r} b_r:, \tag{24}$$

generate a Virasoro algebra with central charge $c = 1/2$ under the Dirac bracket (22).

4 Conclusion

It has been shown that the theory of constrained Hamiltonian systems can be applied to reduce Fock representations of the Virasoro algebra. The explicit realizations (15) and (24) are of course well known, but the unconstrained companion of the Feigin-Fuks representation (5) (with non-zero λ) may be new. The true value of the method is that it can be applied to more complicated algebras, such as diffeomorphism and current algebras in higher dimensions. In that case it can be much more natural to describe a representation as a Fock module plus constraints, rather than to explicitly solve the constraints [4].

It appears that the examples reflect the spin-statistics theorem. Namely, it follows from (12) that (4) can only hold for bosons, and from (22) that (20) can only hold for fermions (otherwise $C_{rs} = 0$). Thus, for bosons (fermions) the canonical momentum is proportional to the first (zeroth) time derivative of the field, which is typical for integer (half-integer) spin.

References

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